Prof. Amador Martin-Pizarro
Exercises: Charlotte Amann-Bartnick

Albert-Ludwigs-Universität Freiburg
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Model Theory

Sheet 3

Deadline: 06.11.25, 2:30 pm.

Exercise 1 (4 points).

Given an $\mathcal{L}$-structure $\mathcal{A}$, choose for every $\mathcal{L}$-formula $\varphi[x_1, \ldots, x_n]$ a new n-ary relation-symbol R_φ. Denote by $\mathcal{L}'$ the expansion of $\mathcal{L}$ by these relation-symbols. The structure $\mathcal{A}$ is now viewed as an $\mathcal{L}'$-structure $\mathcal{A}'$ as follows: The tuple $(a_1, \ldots, a_n)$ lies in $R_\varphi^{\mathcal{A}'}$ if and only if $\mathcal{A} \models \varphi[a_1, \ldots, a_n]$.

a) Show by induction on the complexity of the formula that for every $\mathcal{L}'$-formula $\psi[x_1, \ldots, x_n]$ there exists an $\mathcal{L}$-formula $\theta[x_1, \ldots, x_n]$ such that $\mathcal{A}' \models \forall \bar{x}(\psi[\bar{x}] \leftrightarrow \theta[\bar{x}])$.

b) Does the theory $\mathrm{Th}(\mathcal{A}')$ have quantifier elimination?

Exercise 2 (8 points).

In the language $\mathcal{L} = \{0, s\}$ with a unary function symbol s let $\mathcal{Z} = (\mathbb{Z}, 0^{\mathcal{Z}}, s^{\mathcal{Z}})$ be the $\mathcal{L}$-structure where $s^{\mathcal{Z}}(x) = x + 1$ is interpreted as the successor function. Note that $s^{\mathcal{Z}}$ satisfies the following properties: s is bijective (χ_0) and for all $n > 0$ there is no element x such that $s^n(x) = x$, where s^n denotes the n-fold composition (χ_n).

a) The following eqivalence relation can be defined on models of $\mathrm{Th}(\mathcal{Z})$: Two elements x and y are related if there exists a natural number n from $\mathbb{N}$ such that $s^n(x) = y$ or $s^n(y) = x$ (where s^0 denotes the identity map).

   Show that there is an elementary extension $\mathcal{Z}'$ of $\mathcal{Z}$ with an element that is not related to $0^{\mathcal{Z}}$.

b) Using the ideas of a), show that the collection $\{\chi_n \mid n \in \mathbb{N}\}$ (formulated as $\mathcal{L}$-sentences) is complete with quantifier elimination. Conclude that $\mathrm{Th}(\mathcal{Z})$ has an explicit axiomatization.

Exercise 3 (3 points).

In the language $\mathcal{L} = \{<\}$ consider the $\mathcal{L}$-structure $\mathcal{R}$ with universe $\mathbb{R}$ and the canonical linear order. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Every element from $\mathbb{R}$ can be identified with an element in the ultrapower $\mathcal{R}^{\mathcal{U}}$, whose universe is $\prod_{\mathcal{U}} \mathbb{R}$, since $\mathcal{R}^{\mathcal{U}}$ is an elementary extension of $\mathcal{R}$.

Show that there exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $\mathcal{R}^{\mathcal{U}}$ such that $0 <^{\mathcal{R}^{\mathcal{U}}} \zeta_{n+1} <^{\mathcal{R}^{\mathcal{U}}} \zeta_n <^{\mathcal{R}^{\mathcal{U}}} r$ for every element $r > 0$ from $\mathbb{R}$.

Exercise 4 (5 points).

Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $C \subset B \subset A$. Denote by $S_n^{\mathcal{A}}(B)$ the set of all complete n-types in $\mathcal{A}$ over B and consider the map $\mathrm{restr} : S_n^{\mathcal{A}}(B) \to S_n^{\mathcal{A}}(C)$ given by restriction: For $\bar{c}$ in C and an $\mathcal{L}$-formula $\phi[\bar{x}, \bar{y}]$, the instance $\phi[\bar{x}, \bar{c}]$ is in $\mathrm{restr}(p)$ if $\phi[\bar{x}, \bar{c}]$ (viewed as a B-instance) is in p.

a) Show that the map restr is well-defined.

b) Using Zorn's Lemma show that restr is surjective.

The exercise sheets can be handed in in pairs. Submit them in the mailbox 3.19 in the basement of the Mathematical Institute.